Objectives:

- Determine the response form of the circuit
- Natural response parallel RLC circuits
- Natural response series RLC circuits
- Step response of parallel and series RLC circuits
Natural Response of Parallel RLC Circuits

The problem – given initial energy stored in the inductor and/or capacitor, find \( v(t) \) for \( t \geq 0 \).
It is convenient to calculate $v(t)$ for this circuit because

A. The voltage must be continuous for all time

B. The voltage is the same for all three components

C. Once we have the voltage, it is pretty easy to calculate the branch current

D. All of the above
Natural Response of Parallel RLC Circuits

The problem – given initial energy stored in the inductor and/or capacitor, find $v(t)$ for $t \geq 0$.

KCL: \[
C \frac{dv(t)}{dt} + \frac{1}{L} \int_0^t v(x)dx + I_0 \frac{v(t)}{R} = 0
\]

Differentiate both sides to remove the integral:

\[
C \frac{d^2 v(t)}{dt^2} + \frac{1}{L} v(t) + \frac{1}{R} \frac{dv(t)}{dt} = 0
\]

Divide both sides by $C$ to place in standard form:

\[
\frac{d^2 v(t)}{dt^2} + \frac{1}{LC} v(t) + \frac{1}{RC} \frac{dv(t)}{dt} = 0
\]
The problem – given initial energy stored in the inductor and/or capacitor, find $v(t)$ for $t \geq 0$.

Describing equation:

$$\frac{d^2 v(t)}{dt^2} + \frac{1}{LC} v(t) + \frac{1}{RC} \frac{dv(t)}{dt} = 0$$

This equation is

- Second order
- Homogeneous
- Ordinary differential equation
- With constant coefficients
Once again we want to pick a possible solution to this differential equation. This must be a function whose first AND second derivatives have the same form as the original function, so a possible candidate is

- A. $K \sin \omega t$
- B. $Ke^{-at}$
- C. $Kt^2$
Natural Response of Parallel RLC Circuits

The problem – given initial energy stored in the inductor and/or capacitor, find $v(t)$ for $t \geq 0$.

Describing equation:

$$\frac{d^2 v(t)}{dt^2} + \frac{1}{LC} v(t) + \frac{1}{RC} \frac{dv(t)}{dt} = 0$$

The circuit has two initial conditions that must be satisfied, so the solution for $v(t)$ must have two constants. Use $v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \ V$; Substitute:

$$(s_1^2 A_1 e^{s_1 t} + s_2^2 A_2 e^{s_2 t}) + \frac{1}{RC} (s_1 A_1 e^{s_1 t} + s_2 A_2 e^{s_2 t}) + \frac{1}{LC} (A_1 e^{s_1 t} + A_2 e^{s_2 t}) = 0$$

$$\Rightarrow \quad [s_1^2 + (1/RC)s_1 + (1/LC)]A_1 e^{s_1 t} + [s_2^2 + (1/RC)s_2 + (1/LC)]A_2 e^{s_2 t} = 0$$
Natural Response of Parallel RLC Circuits

The problem – given initial energy stored in the inductor and/or capacitor, find \( v(t) \) for \( t \geq 0 \).

Describing equation:

\[
\frac{d^2v(t)}{dt^2} + \frac{1}{LC}v(t) + \frac{1}{RC} \frac{dv(t)}{dt} = 0
\]

Solution:

\[ v(t) = A_1e^{s_1t} + A_2e^{s_2t} \]

Where \( s_1 \) and \( s_2 \) are solutions for the CHARACTERISTIC EQUATION:

\[ s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \]
\[ s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \]

is called the “characteristic equation” because it characterizes the circuit.

**A. True**

**B. False**
Natural Response of Parallel RLC Circuits

The problem – given initial energy stored in the inductor and/or capacitor, find $v(t)$ for $t \geq 0$.

The two solutions to the characteristic equation can be calculated using the quadratic formula:

$$s^2 + \left(\frac{1}{RC}\right)s + \left(\frac{1}{LC}\right) = 0; \quad s_{1,2} = \frac{-\left(\frac{1}{RC}\right) \pm \sqrt{\left(\frac{1}{RC}\right)^2 - 4\left(\frac{1}{LC}\right)}}{2}$$

$$s_{1,2} = -\left(\frac{1}{2RC}\right) \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \left(\frac{1}{LC}\right)} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

where $\alpha = \frac{1}{2RC}$ (the neper frequency in rad/s)

and $\omega_0 = \frac{1}{\sqrt{LC}}$ (the resonant radian frequency in rad/s)
So far, we know that the parallel RLC natural response is given by

\[ v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \]

\[ s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega^2_0} \quad \text{where} \quad \alpha = \frac{1}{2RC} \quad \text{and} \quad \omega_0 = \sqrt{\frac{1}{LC}} \]

There are three different forms for \( s_1 \) and \( s_2 \). For a parallel RLC circuit with specific values of \( R \), \( L \) and \( C \), the form for \( s_1 \) and \( s_2 \) depends on

- A. The value of \( \alpha \)
- B. The value of \( \omega_0 \)
- C. The value of \( (\alpha^2 - \omega^2_0) \)

C. The value of \( (\alpha^2 - \omega^2_0) \)
**Natural Response – Overdamped Example**

Given \( V_0 = 12 \text{ V} \) and \( I_0 = 30 \text{ mA} \), find \( v(t) \) for \( t \geq 0 \).

\[
\alpha = \frac{1}{2RC} = \frac{1}{2(200)(0.2\mu)} = 12,500 \text{ rad/s}
\]

\[
\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(0.05)(0.2\mu)}} = 10,000 \text{ rad/s}
\]

\( \alpha^2 > \omega_0^2 \) so this is the overdamped case!

\[
s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -12,500 \pm \sqrt{(12,500)^2 - (10,000)^2}
\]

\[
= -12,000 \pm 7500 \quad \Rightarrow \quad s_1 = -5000 \text{ rad/s}, \ s_2 = -20,000 \text{ rad/s}
\]
Natural Response – Overdamped Example

Given $V_0 = 12$ V and $I_0 = 30$ mA, find $v(t)$ for $t \geq 0$.

$$v(t) = A_1 e^{-5000t} + A_2 e^{-20,000t} \text{ V, for } t \geq 0$$

Now we must use the coefficients in the equation to satisfy the initial conditions in the circuit:

$$v(t) \bigg|_{t=0} \text{ in the equation} = v(t) \bigg|_{t=0} \text{ in the circuit}$$

$$\frac{dv(t)}{dt} \bigg|_{t=0} \text{ in the equation} = \frac{dv(t)}{dt} \bigg|_{t=0} \text{ in the circuit}$$
Natural Response – Overdamped Example

Given $V_0 = 12$ V and $I_0 = 30$ mA, find $v(t)$ for $t \geq 0$.

Equation:  
$$v(0) = A_1 e^{-5000(0)} + A_2 e^{-20,000(0)} = A_1 + A_2$$

Circuit:  
$$v(0) = V_0 = 12 \text{ V}$$

$$\Rightarrow \quad A_1 + A_2 = 12$$

Equation:  
$$\frac{dv(0)}{dt} = (-5000)A_1 e^{-5000(0)} + (-20,000)A_2 e^{-20,000(0)}$$

$$= -5000A_1 - 20,000A_2$$
Now we need the initial value of the first derivative of the voltage from the circuit. The describing equation of which circuit component involves $dv(t)/dt$?

- A. The resistor
- B. The inductor
- C. The capacitor
Natural Response – Overdamped Example

Given $V_0 = 12 \text{ V}$ and $I_0 = 30 \text{ mA}$, find $v(t)$ for $t \geq 0$.

Equation: 

$$\frac{dv(0)}{dt} = (-5000)A_1 e^{-5000(0)} + (-20,000)A_2 e^{-20,000(0)}$$

$$= -5000A_1 - 20,000A_2$$

Circuit: 

$$i_C(t) = C \frac{dv(t)}{dt} \quad \Rightarrow \quad \frac{dv(0)}{dt} = \frac{1}{C} i_C(0) = \frac{1}{C} (-i_L(0) - i_R(0))$$

$$\frac{dv(0)}{dt} = \frac{1}{C} \left( -i_L(0) - \frac{v(0)}{R} \right) = \frac{1}{0.2 \mu} \left( -0.03 - \frac{12}{200} \right) = -450,000 \text{ V/s}$$

$$\Rightarrow -5000A_1 - 20,000A_2 = -450,000$$
Natural Response – Overdamped Example

Given $V_0 = 12 \, \text{V}$ and $I_0 = 30 \, \text{mA}$, find $v(t)$ for $t \geq 0$.

\[ v(t) = A_1 e^{-5000t} + A_2 e^{-20,000t} \, \text{V}, \text{ for } t \geq 0 \]

\[ A_1 + A_2 = 12; \quad -5000A_1 - 20,000A_2 = -450,000 \]

Solving simultaneously, \( A_1 = -14, \quad A_2 = 26 \)

Thus,

\[ v(t) = -14e^{-5000t} + 26e^{-20,000t} \, \text{V}, \text{ for } t \geq 0 \]

Checks:

\[ v(0) = -14 + 26 = 12 \, \text{V} \quad (\text{OK}) \]

\[ v(\infty) = 0 \quad (\text{OK}) \]
Given $V_0 = 12$ V and $I_0 = 30$ mA, find $v(t)$ for $t \geq 0$.

You can solve this problem using the Second-Order Circuits table:

1. Make sure you are on the Natural Response side.
2. Find the parallel RLC column.
3. Use the equations in Row 4 to calculate $\alpha$ and $\omega_0$.
4. Compare the values of $\alpha$ and $\omega_0$ to determine the response form (given in one of the last 3 rows).
5. Use the equations to solve for the unknown coefficients.
6. Write the equation for $v(t), t \geq 0$.
7. Solve for any other quantities requested in the problem.
Given $V_0 = 12 \text{ V}$ and $I_0 = 30 \text{ mA}$, find $v(t)$ for $t \geq 0$.

You can solve this problem using the Second-Order Circuits table:

1. Make sure you are on the Natural Response side.
2. Find the parallel RLC column.
3. Use the equations in Row 4 to calculate $\alpha$ and $\omega_0$.
4. Compare the values of $\alpha$ and $\omega_0$ to determine the response form (given in one of the last 3 rows).
5. Use the equations to solve for the unknown coefficients.
6. Write the equation for $v(t)$, $t \geq 0$.
7. Solve for any other quantities requested in the problem.
The values of the ___________ determine whether the response is overdamped, underdamped, or critically damped

A. Initial conditions
B. R, L, and C components
C. Independent sources
Natural Response of Parallel RLC Circuits

The problem – given initial energy stored in the inductor and/or capacitor, find \( v(t) \) for \( t \geq 0 \).

Recap:

\[
s^2 + \left(\frac{1}{RC}\right)s + \left(\frac{1}{LC}\right) = 0; \quad s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}; \quad \alpha = \frac{1}{2RC}; \quad \omega_0 = \sqrt{\frac{1}{LC}}
\]

\( \alpha^2 > \omega_0^2 \): overdamped, so \( v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \)

\( \alpha^2 < \omega_0^2 \): underdamped, so \( s_{1,2} = -\alpha \pm j\omega_d \) where \( \omega_d = \sqrt{\omega_0^2 - \alpha^2} \)

\[
v(t) = A_1 e^{(-\alpha + j\omega_d)t} + A_2 e^{(-\alpha - j\omega_d)t} = A_1 e^{-\alpha t} e^{j\omega_d t} + A_2 e^{-\alpha t} e^{-j\omega_d t}
\]

Note Euler's identity: \( e^{jx} = \cos x + j \sin x; \quad e^{-jx} = \cos x - j \sin x \)

\[
v(t) = A_1 e^{-\alpha t} (\cos \omega_d t + j \sin \omega_d t) + A_2 e^{-\alpha t} (\cos \omega_d t - j \sin \omega_d t)
\]

\[
= e^{-\alpha t} \cos \omega_d t (A_1 + A_2) + e^{-\alpha t} \sin \omega_d t (jA_1 - jA_2)
\]

\[
\Rightarrow v(t) = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t
\]
When the response is underdamped, the voltage is given by the equation

\[ v(t) = B_1 e^{-\alpha t} \cos(\omega_d t) + B_2 e^{-\alpha t} \sin(\omega_d t) \]

In this equation, the coefficients \( B_1 \) and \( B_2 \) are

- **A.** Real numbers
- **X** B. Imaginary numbers
- **X** C. Complex conjugate numbers
Natural Response – Underdamped Example

Given \( V_0 = 0 \) V and \( I_0 = -12.25 \) mA, find \( v(t) \) for \( t \geq 0 \).

\[
\alpha = \frac{1}{2RC} = \frac{1}{2(20,000)(0.125\mu)} = 200 \text{rad/s}
\]

\[
\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(8)(0.125\mu)}} = 1000 \text{rad/s}
\]

\( \alpha^2 < \omega_0^2 \) so this is the underdamped case!

\[
\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{(1000)^2 - (200)^2} = 979.8 \text{rad/s}
\]

\[
\Rightarrow \quad v(t) = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t
\]

\[
= B_1 e^{-200t} \cos 979.8t + B_2 e^{-200t} \sin 979.8t \text{ V, } t \geq 0
\]
Now we evaluate $v(0)$ and $dv(0)/dt$ from the equation for $v(t)$, and set those values equal to $v(0)$ and $dv(0)/dt$ from the circuit, solving for $B_1$ and $B_2$. The values for $v(0)$ and $dv(0)/dt$ from the circuit do not depend on whether the response is overdamped, underdamped, or critically damped.

A. True
B. False
Natural Response – Underdamped Example

Given $V_0 = 0 \text{ V}$ and $I_0 = -12.25 \text{ mA}$, find $v(t)$ for $t \geq 0$.

Equation: $v(0) = B_1 e^{-200(0)} \cos 979.8(0) + B_2 e^{-200(0)} \sin 979.8(0) = B_1$

Circuit: $v(0) = V_0 = 0 \text{ V}$

$\implies B_1 = 0$
Natural Response – Underdamped Example

Given \( V_0 = 0 \) V and \( I_0 = -12.25 \) mA, find \( v(t) \) for \( t \geq 0 \).

Equation:

\[
\frac{dv(0)}{dt} = (-200)B_1 e^{-200(0)} \cos 979.8(0) - 979.8B_1 e^{-200(0)} \sin 979.8(0) \\
+ (-200)B_2 e^{-200(0)} \sin 979.8(0) + 979.8B_2 e^{-200(0)} \cos 979.8(0) \\
= -\alpha B_1 + \omega_d B_2 = -200B_1 + 979.8B_2
\]

Circuit:

\[
\frac{dv_c(0)}{dt} = \frac{1}{C} i_c(0) = \frac{1}{C} \left( -I_0 - \frac{V_0}{R} \right) \\
= \frac{1}{0.125\mu} \left( -(-0.01225) - \frac{0}{20,000} \right) = 98,000 \text{ V/s}
\]

\[\Rightarrow -200B_1 + 979.8B_2 = 98,000 \text{ V/s}\]
Given $V_0 = 0$ V and $I_0 = -12.25$ mA, find $v(t)$ for $t \geq 0$.

\[ v(t) = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t \]

\[ v(0) = B_1 = V_0 = 0; \]

\[ \frac{dv(0)}{dt} = -\alpha B_1 + \omega_d B_2 = -200B_1 + 979.8B_2 \]

\[ = \frac{1}{C} (-I_0 - V_0/R) = 98,000 \quad \Rightarrow \quad B_2 = 100 \]

\[ \therefore \quad v(t) = 100e^{-200t} \sin 979.8t \text{ V}, \, t \geq 0 \]
Natural Response of Parallel RLC Circuits

The problem – given initial energy stored in the inductor and/or capacitor, find $v(t)$ for $t \geq 0$.

Recap:

$$s^2 + (1/RC)s + (1/LC) = 0; \quad s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}; \quad \alpha = \frac{1}{2RC}; \quad \omega_0 = \sqrt{\frac{1}{LC}}$$

$$\alpha^2 > \omega_0^2 : \quad \text{overdamped, so } v(t) = A_1 e^{s_{1}t} + A_2 e^{s_{2}t}$$

$$\alpha^2 < \omega_0^2 : \quad \text{underdamped, so } v(t) = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t$$

where \[ \omega_d = \sqrt{\omega_0^2 - \alpha^2} \]

$$\alpha^2 = \omega_0^2 : \quad \text{Critically damped, so } s_{1,2} = -\alpha \pm 0 = -\alpha$$
When the response is critically damped, a reasonable expression for the voltage is

\[ v(t) = A_1 e^{-\alpha t} + A_2 e^{-\alpha t} \text{ V, } t \geq 0 \]
Natural Response of Parallel RLC Circuits

The problem – given initial energy stored in the inductor and/or capacitor, find $v(t)$ for $t \geq 0$.

When the circuit’s response is critically damped, the assumed form of the solution we have been using up until now does not provide enough unknown coefficients to satisfy the two initial conditions from the circuit. Therefore, we use a different solution form:

$$\alpha^2 = \omega_0^2 : \text{Critically damped so}$$

$$v(t) = D_1 t e^{s_1 t} + D_2 e^{s_2 t} = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$
Natural Response – Critically damped Example

Given \( V_0 = 50 \text{ V} \) and \( I_0 = 250 \text{ mA} \), find \( v(t) \) for \( t \geq 0 \).

\[
\alpha = \frac{1}{2RC} = \frac{1}{2(100)(10\mu)} = 500 \text{ rad/s}
\]

\[
\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(0.4)(10\mu)}} = 500 \text{ rad/s}
\]

\( \alpha^2 = \omega_0^2 \) so this is the critically damped case!

\[
\Rightarrow \quad v(t) = D_1t e^{-\alpha t} + D_2 e^{-\alpha t} = D_1t e^{-500t} + D_2 e^{-500t} \quad \forall, t \geq 0
\]
Natural Response – Critically damped Example

Given $V_0 = 50$ V and $I_0 = 250$ mA, find $v(t)$ for $t \geq 0$.

Use the initial conditions from the equation and from the circuit to solve for the unknown coefficients.

Equation: \[ v(0) = D_1(0)e^{-500(0)} + D_2e^{-500(0)} = D_2 \]

Circuit: \[ v(0) = V_0 = 50 \text{ V} \quad \Rightarrow \quad D_2 = 50 \]
Natural Response – Critically damped Example

Given $V_0 = 50 \text{ V}$ and $I_0 = 250 \text{ mA}$, find $v(t)$ for $t \geq 0$.

Equation:

$$\frac{dv(0)}{dt} = D_1 e^{-500(0)} + D_1 (-500)(0)e^{-500(0)} + D_2 (-500)e^{-500(0)}$$

$$= D_1 - 500D_2$$

Circuit:

$$\frac{dv_C(0)}{dt} = \frac{1}{C} i_C(0) = \frac{1}{C} \left(-I_0 - \frac{V_0}{R}\right)$$

$$= \frac{1}{10 \mu} \left(-0.25 - \frac{50}{100}\right) = -75,000 \text{ V/s}$$

$$\Rightarrow \quad D_1 - 500D_2 = -75,000 \text{ V/s}$$
Natural Response – Critically damped Example

Given $V_0 = 50$ V and $I_0 = 250$ mA, find $v(t)$ for $t \geq 0$.

$$v(t) = D_1 t e^{-500t} + D_2 e^{-500t}$$
$$v(0) = D_2 = V_0 = 50;$$
$$\frac{dv(0)}{dt} = D_1 - \alpha D_2 = D_1 - 500D_2$$

$$= \frac{1}{C} \left( - I_0 - V_0 / R \right) = -78,000 \quad \Rightarrow \quad D_1 = -50,000$$

$$\therefore \quad v(t) = -50,000te^{-500t} + 50e^{-500t} \text{ V, } t \geq 0$$
Natural Response of Parallel RLC Circuits – Summary

The problem – given initial energy stored in the inductor and/or capacitor, find \( v(t) \) for \( t \geq 0 \).

Use the Second-Order Circuits table:

1. Make sure you are on the Natural Response side.
2. Find the parallel RLC column.
3. Use the equations in Row 4 to calculate \( \alpha \) and \( \omega_0 \).
4. Compare the values of \( \alpha \) and \( \omega_0 \) to determine the response form (given in one of the last 3 rows).
5. Use the equations to solve for the unknown coefficients.
6. Write the equation for \( v(t), t \geq 0 \).
7. Solve for any other quantities requested in the problem.
Step Response of Second-order RLC Circuits

The problem – find the response for $t \geq 0$. Note that there may or may not be initial energy stored in the inductor and capacitor!
The circuit for the parallel RLC step response is repeated here. Consider how this circuit behaves as $t \to \infty$. Which component’s final value is non-zero?

A. The resistor

B. The inductor

C. The capacitor

D. All of the above
The only component whose final value is NOT zero is the inductor, whose final current is the current supplied by the source. We now have to construct a response form that can satisfy two initial conditions and one non-zero final value. We can satisfy the final value directly if we specify the inductor current as the response we will solve for:

\[ i_L(t) = I_F + (\text{the form of the natural response}) \]
Step Response of a Parallel RLC Circuit

The problem – there is no initial energy stored in this circuit; find $i(t)$ for $t \geq 0$.

To begin, find the initial conditions and the final value. The initial conditions for this problem are both zero; the final value is found by analyzing the circuit as $t \to \infty$. 
Step Response of a Parallel RLC Circuit

The problem – there is no initial energy stored in this circuit; find $i(t)$ for $t \geq 0$.

$t \rightarrow \infty$:

$I_F = 24 \text{ mA}$
Step Response of a Parallel RLC Circuit

The problem – there is no initial energy stored in this circuit; find $i(t)$ for $t \geq 0$.

Next, calculate the values of $\alpha$ and $\omega_0$ and determine the form of the response:

\[
\alpha = \frac{1}{2RC} = \frac{1}{2(400)(25\text{n})} = 50,000\text{ rad/s}
\]

\[
\omega_0 = \sqrt{1/LC} = \sqrt{1/(25\text{m})(25\text{n})} = 40,000\text{ rad/s}
\]
We just calculated $\alpha = 50,000 \text{ rad/s}$ and $\omega_0 = 40,000 \text{ rad/s}$, so the form of the response is

- **A.** Overdamped
- **X** B. Underdamped
- **X** C. Critically damped
Once we know the response form is overdamped, we know we have to calculate

- A. $\omega_d$
- B. $s_1$ and $s_2$
- C. Nothing additional
Step Response of a Parallel RLC Circuit

The problem – there is no initial energy stored in this circuit; find \( i(t) \) for \( t \geq 0 \).

Since the response form is overdamped, calculate the values of \( s_1 \) and \( s_2 \):

\[
s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -50,000 \pm \sqrt{50,000^2 - 40,000^2}
\]

\[
= -50,000 \pm 30,000 \text{rad/s}
\]

\[
\therefore s_1 = -20,000 \text{rad/s} \quad \text{and} \quad s_2 = -80,000 \text{rad/s}
\]

\[
\Rightarrow \quad i_L(t) = 0.024 + A_1 e^{-20,000t} + A_2 e^{-80,000t} \quad \text{A, } t \geq 0
\]
Step Response of a Parallel RLC Circuit

The problem – there is no initial energy stored in this circuit; find $i(t)$ for $t \geq 0$.

$$i_L(t) = 0.024 + A_1 e^{-20,000 \alpha} + A_2 e^{-80,000 \alpha} \text{ A, } t \geq 0$$

Next, set the values of $i(0)$ and $di(0)/dt$ from the equation equal to the values of $i(0)$ and $di(0)/dt$ from the circuit.

From the equation: $$i_L(0) = 0.024 + A_1 + A_2$$

From the circuit: $$i_L(0) = I_0 = 0$$
Step Response of a Parallel RLC Circuit

The problem – there is no initial energy stored in this circuit; find \( i(t) \) for \( t \geq 0 \).

\[
i_L(t) = 0.024 + A_1 e^{-20,000 \alpha} + A_2 e^{-80,000 \alpha} \text{ A, } t \geq 0
\]

From the equation:

\[
\frac{di_L(0)}{dt} = -20,000A_1 - 80,000A_2
\]

From the circuit:

\[
\frac{di_L(0)}{dt} = \frac{v_L(0)}{L} = \frac{V_0}{L} = 0
\]
Step Response of a Parallel RLC Circuit

The problem – there is no initial energy stored in this circuit; find \( i(t) \) for \( t \geq 0 \).

\[
i_L(t) = 0.024 + A_1 e^{-20,000 \alpha} + A_2 e^{-80,000 \alpha} \text{ A, } t \geq 0
\]

Solve: \[ 0.024 + A_1 + A_2 = 0 \]
and \[ -20,000 A_1 - 80,000 A_2 = 0 \]

:.
\[
A_1 = -32 \text{ mA}; \quad A_2 = 8 \text{ mA}
\]

\[
\Rightarrow \quad i_L(t) = 24 - 32e^{-20,000 \alpha} + 8e^{-80,000 \alpha} \text{ mA, } t \geq 0
\]
We can check this result at $t = 0$ and as $t \to \infty$; from the equation we get

\[ i_L(t) = 24 - 32e^{-20,000t} + 8e^{-80,000t} \text{ mA, } t \geq 0 \]

We can check this result at $t = 0$ and as $t \to \infty$; from the equation we get

- A. $i_L(0) = 32 \text{ mA, } i_L(\infty) = 0$
- B. $i_L(0) = 0, \quad i_L(\infty) = 0$
- C. $i_L(0) = 0, \quad i_L(\infty) = 24 \text{ mA}$
If we now want to find $v(t)$ for $t \geq 0$, we need to

- ✔️ A. Find the derivative of the current and multiply by $L$
- ❌ B. Find the integral of the current and divide by $C$
- ❌ C. Multiply the current by $R$
Step Response of RLC Circuits – Summary

Use the Second-Order Circuits table:

1. Make sure you are on the Step Response side.
2. Find the appropriate column for the RLC circuit topology.
3. Find the values of the two initial conditions and the one non-zero final value. Make sure the initial conditions and final value are defined exactly as shown in the figure!
4. Use the equations in Row 4 to calculate $\alpha$ and $\omega_0$.
5. Compare the values of $\alpha$ and $\omega_0$ to determine the response form (given in one of the last 3 rows).
6. Write the equation for $v_C(t)$, $t \geq 0$ (series) or $i_L(t)$, $t \geq 0$ (parallel), leaving only the 2 coefficients unspecified.
7. Use the equations provided to solve for the unknown coefficients.
8. Solve for any other quantities requested in the problem.
The problem – find \( v_c(t) \) for \( t \geq 0 \).

Find the initial conditions by analyzing the circuit for \( t < 0 \):

\[
V_0 = \frac{15k}{9k + 15k}(80 \text{ V}) = 50 \text{ V}
\]

\[ I_0 = 0 \text{ A} \]
Step Response of a Series RLC Circuit

The problem – find $v_C(t)$ for $t \geq 0$.

Find the final value of the capacitor voltage by analyzing the circuit as $t \to \infty$:

$V_F = 100$ V
Step Response of a Series RLC Circuit

The problem – find $v_c(t)$ for $t \geq 0$.

Use the circuit for $t \geq 0$ to find the values of $\alpha$ and $\omega_0$:

$$\alpha = \frac{R}{2L} = \frac{80}{2(0.005)} = 8000\text{rad/s}$$

$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(0.005)(2\mu)}}$$

$$= 10,000\text{rad/s}$$

$$\alpha^2 < \omega_0^2 \quad \Rightarrow \quad \text{underdamped}$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{10,000^2 - 8000^2}$$

$$= 6000\text{rad/s}$$
Step Response of a Series RLC Circuit

The problem – find $v_C(t)$ for $t \geq 0$.

Write the equation for the response and solve for the unknown coefficients:

$$v_C(t) = 100 + B_1 e^{-8000t} \cos 6000t + B_2 e^{-8000t} \sin 6000t \text{ V, } t \geq 0$$

$$v_C(0) = V_F + B_1 = V_0 \quad \therefore \quad 100 + B_1 = 50$$

$$\frac{dv_C(0)}{dt} = -\alpha B_1 + \omega_d B_2 = \frac{I_0}{C} \quad \therefore \quad -8000B_1 + 6000B_2 = 0$$

$$\Rightarrow \quad B_1 = -50 \text{ V, } \quad B_2 = 66.67 \text{ V}$$

$$v_C(t) = 100 - 50e^{-8000t} \cos 6000t + 66.67e^{-8000t} \sin 6000t \text{ V, } t \geq 0$$
Natural Response of Series RLC Circuits

The problem – given initial energy stored in the inductor and/or capacitor, find $i(t)$ for $t \geq 0$. 
The difference(s) between the analysis of series RLC circuit and the parallel RLC circuit is/are:

- **X** A. The variable we calculate.
- **X** B. The describing differential equation.
- **X** C. The equations for satisfying the initial conditions
- **√** D. All of the above
Natural Response of Series RLC Circuits

The problem – given initial energy stored in the inductor and/or capacitor, find $i(t)$ for $t \geq 0$.

KVL: \[ L \frac{di(t)}{dt} + \frac{1}{C} \int_0^i i(x) dx + V_0 + Ri(t) = 0 \]

Differentiate both sides to remove the integral:

\[ L \frac{d^2i(t)}{dt^2} + \frac{1}{C} i(t) + R \frac{di(t)}{dt} = 0 \]

Divide both sides by $L$ to place in standard form:

\[ \frac{d^2i(t)}{dt^2} + \frac{R}{L} \frac{di(t)}{dt} + \frac{1}{LC} i(t) = 0 \]
The describing differential equation for the series RLC circuit is

\[ \frac{d^2 i(t)}{dt^2} + \frac{R}{L} \frac{di(t)}{dt} + \frac{1}{LC} i(t) = 0 \]

Therefore, the characteristic equation is

A. \[ s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \]

B. \[ s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \]

C. \[ s^2 + \frac{1}{LC}s + \frac{1}{RC} = 0 \]
Natural Response of Series RLC Circuits

The problem – given initial energy stored in the inductor and/or capacitor, find $i(t)$ for $t \geq 0$.

The two solutions to the characteristic equation can be calculated using the quadratic formula:

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0; \quad s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

where $\alpha = \frac{R}{2L}$ (the neper frequency in rad/s)

and $\omega_0 = \sqrt{\frac{1}{LC}}$ (the resonant radian frequency in rad/s)
Natural Response Series RLC Problems

The problem – given initial energy stored in the inductor and/or capacitor, find $i(t)$ for $t \geq 0$.

You can solve these problem using the Second-Order Circuits table:

1. Make sure you are on the Natural Response side.
2. Find the series RLC column.
3. Use the equations in Row 4 to calculate $\alpha$ and $\omega_0$.
4. Compare the values of $\alpha$ and $\omega_0$ to determine the response form (given in one of the last 3 rows).
5. Use the equations to solve for the unknown coefficients.
6. Write the equation for $i(t)$, $t \geq 0$.
7. Solve for any other quantities requested in the problem.
Natural Response Series RLC Example

The capacitor is charged to 100 V and at \( t = 0 \), the switch closes. Find \( i(t) \) for \( t \geq 0 \).

\[
\alpha = \frac{R}{2L} = \frac{560}{2(0.1)} = 2800 \text{rad/s}
\]

\[
\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(0.1)(0.1 \mu)}} = 10,000 \text{rad/s}
\]

\( \alpha^2 < \omega_0^2 \) so this is the underdamped case!

\[
\omega_d = \sqrt{\omega_0^2 - \alpha^2} = 9600 \text{rad/s}
\]

\[
i(t) = B_1 e^{-2800t} \cos 9600t + B_2 e^{-2800t} \sin 9600t \text{ A, } t \geq 0
\]
Natural Response Series RLC Example

The capacitor is charged to 100 V and at \( t = 0 \), the switch closes. Find \( i(t) \) for \( t \geq 0 \).

\[
i(t) = B_1 e^{-2800t} \cos 9600t + B_2 e^{-2800t} \sin 9600t \text{ A, } t \geq 0
\]

Now we must use the coefficients in the equation to satisfy the initial conditions in the circuit:

\[
i(t) \bigg|_{t=0} \text{ in the equation } = i(t) \bigg|_{t=0} \text{ in the circuit}
\]

\[
\frac{di(t)}{dt} \bigg|_{t=0} \text{ in the equation } = \frac{di(t)}{dt} \bigg|_{t=0} \text{ in the circuit}
\]
The following quantities used to calculate the unknown coefficients are defined by different equations in both the series and parallel RLC natural response problems:

X A. The initial values of voltage or current from the equation.
X B. The initial values of voltage or current from the circuit.
X C. The initial values of the derivative of voltage or current from the equation.
 ✓ D. The initial values of the derivative of voltage or current from the circuit.
Natural Response Series RLC Example

The capacitor is charged to 100 V and at $t = 0$, the switch closes. Find $i(t)$ for $t \geq 0$.

\[ i(t) = B_1 e^{-2800t} \cos 9600t + B_2 e^{-2800t} \sin 9600t \text{ A, } t \geq 0 \]

Equation: \[ i(0) = B_1 \] (same as the parallel case!)

Circuit: \[ i(0) = I_0 = 0 \]

\[ \Rightarrow B_1 = 0 \]
The capacitor is charged to 100 V and at \( t = 0 \), the switch closes. Find \( i(t) \) for \( t \geq 0 \).

Equation:
\[
\frac{di(0)}{dt} = -\alpha B_1 + \omega_d B_2 \quad \text{(same as the parallel case!)}
\]

Circuit:
\[
\frac{di(0)}{dt} = \frac{1}{L} v_L(0) = \frac{1}{L} \left( -v_C(0) - v_R(0) \right) = \frac{1}{L} \left( -V_0 - RI_0 \right)
\]
\[
= \frac{1}{0.1} \left( -(-100) - 560(0) \right) = 1000 \text{ A/s}
\]
\[
\Rightarrow -2800(0) + 9600B_2 = 1000 \quad \Rightarrow \quad B_2 = 0.104
\]
\[
\therefore \quad i(t) = 0.104e^{-2800t} \sin 9600t \text{ A, } t \geq 0
\]
Use the Second-Order Circuits table:

1. Make sure you are on the Natural Response side.
2. Find the appropriate column for the RLC circuit topology.
3. Make sure the initial conditions are defined exactly as shown in the figure!
4. Use the equations in Row 4 to calculate $\alpha$ and $\omega_0$.
5. Compare the values of $\alpha$ and $\omega_0$ to determine the response form (given in one of the last 3 rows).
6. Use the equations to solve for the unknown coefficients.
7. Write the equation for $v(t), \ t \geq 0$ (parallel) or $i(t), \ t \geq 0$ (series).
8. Solve for any other quantities requested in the problem.