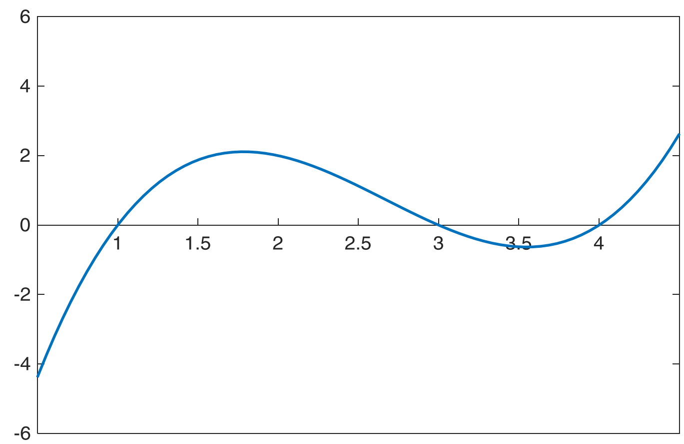
1. Answer and justify (45 %)

1.1 What is machine precision? **(4)**

1.2 Find the absolute error and the relative error if X=3.162 is used to approximate $x=\sqrt{10}$. Find the relative error of X^4. **(4)**

1.3 Show that the equation $x = 1 – x^4$ has a root in the interval [0.7 0.8]! **(4)**

1.4 The **bisection method** is applied to the function *f(x) = (x − 1)(x − 3)(x − 4)* (graph given to the right). The initial interval used for the method is [a, b] = [0.5, 4.5]. Explain to which root will the method converge if it converges? **(4)**



1.5 What happens to the truncation error of the composite Simpsons rule when the size of the intervals ***h*** is halved? **(4)**

1.6 Compute the polynomial that interpolates the function *f(x) = sin(x)* at the following points: x = 0, x = π/4 and x = π/2. Find the integral of the polynomial using composite trapeze rule in the three points and sketch the result. Compare with the analytical solution! **(10)**

1.7 Give a MATLAB statement that could be used to compute all of the zeros of the polynomial **(5)**

$P(x) = 3x^5 + 5x^4 – 33x^3 + x - 48$

1.8 To find square root of a positive real number A using Newton-Raphson's method, the equation $x^2 – A = 0$ is iteratively solved where the iterates are **(5)**

$x_{n+1} = x_n – (x_n^2-A) / 2x_n$

How would you find cubic root of a positive number A? Write the iterative

$x_{n+1} = a(x_n) / b(x_n)$

using only two polynomials a(x) and b(x).

1.9 The function *f(x) = ln(x)* is evaluated at *f*(1) = 0, *f*(2) = 0.6931 and *f*(3) = 1.0986. The integral

$I = \int_1^3 f(x)dx$ is approximated by trapeze formula. Explain whether the result will overestimate or underestimate the actual value of the integral! Justify by e.g. using a sketch of the function. **(5)**

1.10 What are the biggest and smallest absolute value non-zero floating point numbers ***A*** and ***B*** one can write using 7 bits: 1 sign bit, 4 mantissa bits and 2 exponent bits (the exponent can have value from -2 to 1). Floating point number = sign * mantissa * 2^exponent, where mantissa = 1. x x **Express A and B as decimal numbers**! **(5)**

1.11 What are round-off and truncation errors? Explain in your words and possibly with some examples! **(6)**

1.12 When finding zero of a continuous differentiable monotonous function with one root, **using bisection method**, convergence is **(4)**

a) Linear,
b) Superlinear,
c) Quadratic,
d) The method is not robust and there may be no convergence.

1.13 In data-fitting, it is the goal to fit the data-set (x_k, y_k) with the function *F(x)*. The method of least squares finds the function *F(x)* so that **(5)**

a) $\sum_{k=0}^{m} F(x_k)^2$ is minimized,
b) $\sum_{k=0}^{m} (y_k - F(x_k))^2$ is minimized,
c) *F(x)* is a quadratic function that passes all points (x_k, y_k),
d) the data-set is integrated by composite Simpson's rule.

1.14 The function *f*(x) is evaluated at *f*(1) = 2, *f*(2) = 2 and *f*(3) = 3. Imagine the integral $I = \int_1^3 f(x)dx$ is approximated by: trapeze formula, composite trapeze formula and Simpson's 1/3 formula. Which approximation gives the **biggest result** and which approximation gives **smallest result**? Justify by e.g. using a plot of the solutions. **(5)**

1.15 What does quadratic convergence of a method mean? Name a method that sometimes has quadratic convergence! **(6)**

1.16 Dataset you have contains 5 points (x, y): (0, 1), (1, 1), (2, 2), (3, 3) and (4, 4). **Write** a formula for the Lagrange polynomial that passes through these points! (Do not reduce the polynomial to its simplest form, just show how to find it). **What** is the degree of the polynomial? **Do** you think it will represent the data well? **Can** you use the Lagrange polynomial for extrapolation of data? **(15)**

1.17 Compare trapezoidal integration to 1/3 Simpsons integration rule! What are advantages of which method? **(8)**

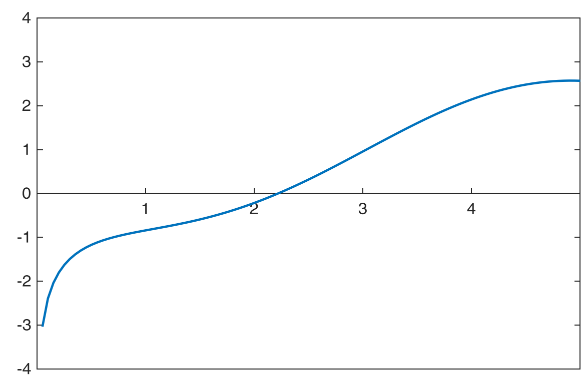
1.18 Explain why it is smart to use the number ϕ in golden section optimization! **(6)**

1.19 What is **FLOPS**? **(5)**

2. General (30%)

2.1 Using **Newton-Raphson's** method, **write** the first three iterations for finding zero of the function $f(x) = \ln(x) - \sin(x)$ shown in the figure! Use $x_0 = 3$. **(20)**

Number of iteration *i*	Guess x_i	$f(x_i)$	$f(x_i)$- $f(x_{i+1})$
0	**3**		-------
1			
2			
3			



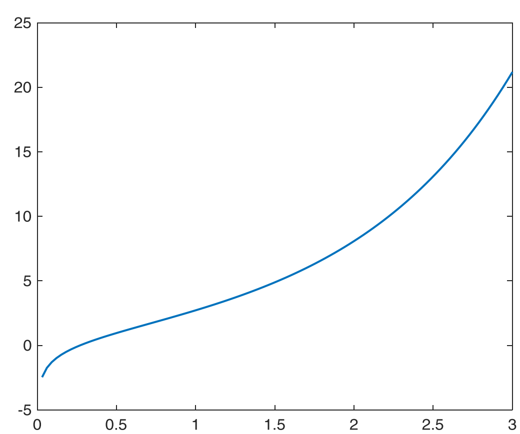
2.2 Write down the **composite** trapeze formula and compute the integral **(10)**

$$I = \int_1^4 f(x)dx$$ if the domain is split in **six equal intervals**.

Find ***I*** if $f(x) = \ln(x) - \sin(x)$

2.3 Using **Newton-Raphson's** method, **write** the first three iterations for finding zero of the function $f(x) = e^x + \ln(x)$ shown in the figure! Use $x_0 = 1$. **(20)**

Number of iteration *i*	Guess x_i	$f(x_i)$	$f(x_i)$- $f(x_{i+1})$
0	**1**	**2.7183**	-------
1			
2			
3			



Considering the calculated error, can you confirm that in this case Newton-Raphson's method has quadratic convergence? **(5)**

2.4 Write down the **composite** trapeze formula and compute the integral **(10)**

$I = \int_1^3 f(x)dx$ if the domain is split in **four** equal intervals. Find ***I*** if $f(x) = e^x + \ln(x)$

2.5 Using Newton-Raphson's method, **write** at least the first two iterations for the equation below! Use x_0 = 0.5. You need the **smallest positive** *x* that satisfies the equation. **Show** graphically what would be a bad initial guess for this method! **(20)**

$$\sin(x) = \frac{\pi}{6}$$

Number of iteration	Guess	Error
0	0.5	---
1		
2		
3		

3. Matlab (25%)

3.1 If the matrix A is defined as **(4)**

```
A=[2 4; 7 9];
```

What is the result of

```
a) A.*A
b) A * A
```

3.2 Consider the function shown in problem 2.

a) The function $f(x) = \ln(x) - \sin(x)$ is sampled in seven equidistant points from x=1 to x=4. Write a Matlab **program** that **approximates** the given function with a **third-order** polynomial ***p***, **(6)**

```
x=linspace(1, 4, 7);
```

```
f=_______________________________;

p=_______________________________;
```

and **program lines** to plot the approximation in the interval using 50 equidistant points!

b) Write a MATLAB **program line** that finds the root of the function $f(x) = \ln(x) - \sin(x)$, in the interval 1 < x < 4. In addition, write a MATLAB **program line** that finds the root of the polynomial ***p*** in approximation from above 3.2a). **(6)**

c) Write a Matlab **program line** that interpolates the data in the given interval by a cubic spline! How do you evaluate the spline at `x=2.1` (one more program line)? **(4)**

3.3 Explain how to solve the second order ordinary differential equation: **(5)**

$$\frac{d^2y}{dt^2} + y = 0$$

With initial conditions y(0) = 5 and y'(0) = 4, in the time ranging from 0 to 10.

3.4 Consider the function shown in problem 2.

d) The function $f(x) = e^x + \ln(x)$ is sampled in five equidistant points from x=1 to x=3. Write a Matlab **program** that **approximates** with **third-order** polynomial the given function, **(6)**

```
x=linspace(1, 3, 5);
y=exp(x)+log(x);
```

and **program lines** to plot the approximation in the interval using 50 equidistant points!

e) You get the result of the approximation above as: **(5)**

```
1.3617   -4.2691    8.6069   -2.9877
```

What does it mean?

f) Write a MATLAB **program line** that finds the root of the function $f(x) = e^x + \ln(x)$, in the interval 0.01 < x < 1. In addition, write a MATLAB **program line** that finds the root of the polynomial approximation from above 3.1a). **(7)**

g) Write a Matlab **program line** that interpolates the data in the given interval by a cubic spline! How do you evaluate the spline at `x=1.1` (one more program line)? **(5)**

h) Now you have a function *f(t)* given by **(10)**

$$\boldsymbol{f}(\boldsymbol{t}) = \int_{1}^{t} (\boldsymbol{e}^{x} + \boldsymbol{ln}(\boldsymbol{x}))\boldsymbol{dx}$$

Complete the small program that computes and plots the function in the interval t=1…4 in 100 points!

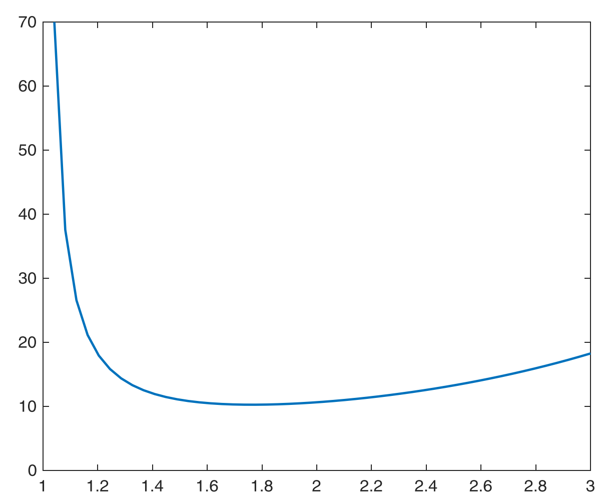
```
t=linspace(1, 4, _______ );
for i=1:100

f(i)=_______________________________________

end

plot( _____ ,______)
```

i) What does the following MATLAB code do: **(7)**

```
fminbnd(@(xx) quad(@(x) exp(x)./log(x), 1.2+xx, 1.4+xx), 0, 2)
```

The function exp(x)/ln(x) is shown to the right.

Is this function unimodal?



3.5 You have a dataset of 7 measurements of solar power collected by a panel, sampled every hour at discrete times t_n with the following values $f(t_n)$:

t [h]	**0**	**1**	**2**	**3**	**4**	**5**	**6**
f(t) [W]	**0.1201**	**4.7509**	**8.4490**	**10.3328**	**9.3699**	**5.8497**	**1.7147**

```
t=[0 1 2 3 4 5 6];
f=[0.1201 4.7509 8.4490 10.3328 9.3699 5.8497 1.7147];
```

a) Write a Matlab program that interpolates a full polynomial to the dataset! **(5)**

b) Write a Matlab program that approximates the data by a 4^{th} degree polynomial! **(5)**

c) Write a Matlab program that interpolates the data by a cubic spline! How do you evaluate, using the spline, the function at `t=3.5` **(5)**

d) It turns out that approximation of the data by trigonometric functions gives best results. The least squares approximation says the function is **(5)**

```
fun=@(tt) 5.7982+1.5216*sin(0.8976*tt)-4.7948*cos(0.8976*tt);
```

How can you approximate the total energy collected by the panel that day? (How do you integrate the function `f(t)`… in Matlab?)

e) How can you estimate after what time was the energy collected 20Wh? You may use the `fzero` command to find this! **(5)**

f) What is the rate of change of solar irradiation 2 hours after the experiment started? Use forward-, backward-, central- **and/or** Richardson formula to get the derivative! This is NOT to be implemented in MATLAB, just find the slope of the function. **(5)**

3.6 A system of coupled ordinary differential equations is given below. The initial conditions are $y_1(0) = 0$, $y_2(0) = -1$. You are interested in values (and behavior) of $y_1(0)$ to $y_1(5)$. How can you solve the system and plot the solution using Matlab? **(10)**

$$\frac{dy_1}{dt} = -4y_1 - 2y_2 + \cos(t)$$

$$\frac{dy_2}{dt} = 3y_1 + y_2$$